M337/Specimen exam 1

Module Examination

Complex analysis

There are **two parts** to this examination.

In **Part 1** you should **submit answers to <u>all</u> 6 questions**. Each question is worth 10% of the total mark.

In **Part 2** you should **submit answers to <u>2 out of the 3</u> questions**. Each question is worth 20% of the total mark.

Do not submit more than two answers for Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Include all your working, as some marks are awarded for this.

Write your answers in **pen**, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work. Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

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Part 1

You should **submit answers to all questions** from Part 1.

Each question is worth 10%.

Question 1

Determine each of the following complex numbers in *Cartesian* form, simplifying your answers as far as possible.

(a) i^{17} [2]

(b) $\dfrac{1+i}{2-i}$ [2]

(c) $\sinh(i\pi/6)$ [2]

(d) $(-8i)^{1/3}$ [4]

Question 2

(a) Let Γ be the line segment from 0 to $1-i$.

(i) Evaluate

$$\int_\Gamma \operatorname{Im} z \, dz. \qquad [3]$$

(ii) Using your answer to part (a)(i), or otherwise, evaluate

$$\int_{\widetilde{\Gamma}} \operatorname{Re}(iz) \, dz, \qquad [2]$$

where $\widetilde{\Gamma}$ is the reverse path of Γ.

(b) Determine an upper estimate for the modulus of

$$\int_C \frac{2\sinh z}{z^5 - 2} \, dz,$$

where C is the circle $\{z : |z| = 2\}$. [5]

Question 3

(a) Locate the poles of the function

$$f(z) = \frac{z}{(3z^2-1)(z^2-3)}$$

and state their orders.

Determine the residues of f at the poles that lie inside the unit circle $C = \{z : |z| = 1\}$. [5]

(b) Hence evaluate the real trigonometric integral

$$\int_0^{2\pi} \frac{1}{1 + 3\sin^2 t} \, dt. \qquad [5]$$

Question 4

Consider the equation

$$z^7 + 5z^3 + 7 = 0.$$

(a) Use Rouché's Theorem to determine the number of solutions of the equation in the annulus $\{z : 1 < |z| < 2\}$. *[8]*

(b) Prove that at least one of the solutions from part (a) is a real number. *[2]*

Question 5

Let q be the velocity function

$$q(z) = \overline{z}^2.$$

(a) Explain why q is the velocity function for an ideal flow on $\mathbb{C}$. *[1]*

(b) Determine a stream function for this flow, and hence find an equation for the streamline through the point $e^{i\pi/3}$. *[4]*

(c) Sketch this streamline and indicate the direction of flow. *[3]*

(d) Find the flux of q across the line segment Γ from $-i$ to i. *[2]*

Question 6

(a) Prove that the iteration sequence

$$z_{n+1} = z_n(1 - z_n), \quad n = 0, 1, 2, \ldots,$$

with $z_0 = \frac{1}{2}$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + \tfrac{1}{4}, \quad n = 0, 1, 2, \ldots,$$

with $w_0 = 0$.

State the conjugating function. *[4]*

(b) Determine whether or not each of the following points c lies in the Mandelbrot set.

(i) $c = \frac{1}{2}i$ *[3]*

(ii) $c = 1 + \frac{1}{2}i$ *[3]*

Part 2

You should **submit answers to two questions** from Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Each question is worth 20%.

Question 7

(a) Let

$$A = \{z : |z - 2i| \leq 1\},$$
$$B = \{iy : y > 0\},$$
$$C = \mathbb{C} - B.$$

(i) Sketch the sets $A \cap B$ and $A \cap C$. *[4]*

(ii) Determine whether or not the function $f(z) = e^{1/z}$ is bounded on each of the sets A, B and C. *[6]*

(b) Use the Cauchy–Riemann Theorem and its converse to determine all the points $z = x + iy$ of $\mathbb{C}$ at which the function

$$f(z) = \cos \overline{z}$$

is differentiable, and find the derivative of f at each of these points. *[10]*

Question 8

(a) Let f be the function

$$f(z) = \frac{4}{z^2 - 4}.$$

Find the Laurent series about 2 for f on each of the following regions.

(i) $A = \{z : 0 < |z - 2| < 4\}$ *[6]*

(ii) $B = \{z : |z - 2| > 4\}$ *[6]*

In each case give four consecutive non-zero terms of the Laurent series.

(b) Let g be the function

$$g(z) = z\cos(1/z^2).$$

(i) Determine the Laurent series about 0 for g, giving three consecutive non-zero terms. *[3]*

(ii) Classify the singularity of g at 0 as a removable singularity, a pole or an essential singularity. Justify your answer. *[2]*

(iii) Prove that there is a complex number z such that

$$\operatorname{Im} g(z) > 1000.$$

[3]

Question 9

Let

$$\mathcal{R} = \{z : |z| < 1,\ \operatorname{Re} z < 0\},$$
$$\mathcal{S} = \{z : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\},$$
$$\mathcal{T} = \{z : \operatorname{Re} z > 0\}.$$

(a) Sketch the regions $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$. [*3*]

(b) Determine a one-to-one conformal mapping f from $\mathcal{R}$ onto $\mathcal{S}$. [*6*]

(c) Determine a one-to-one conformal mapping g from $\mathcal{S}$ onto $\mathcal{T}$. [*4*]

(d) Hence determine a one-to-one conformal mapping h from $\mathcal{R}$ onto $\mathcal{T}$, and find the rule for the inverse function h^{-1}. [*4*]

(e) Find the image of the real line segment $(-1, 0)$ under h. [*3*]

[END OF QUESTION PAPER]